

ON AMALGAMATED BANACH ALGEBRAS

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ABSTRACT. Let A and B be Banach algebras, $\theta : A \rightarrow B$ be a continuous Banach algebra homomorphism and I be a closed ideal in B . Then the direct sum of A and I with respect to θ , denoted $A \bowtie^\theta I$, with a special product becomes a Banach algebra which is called the amalgamated Banach algebra. In this paper, among other things, we compute the topological centre of $A \bowtie^\theta I$ in terms of that of A and I . Using this, we provide a characterization of the Arens regularity of $A \bowtie^\theta I$. Then we determine the character space of $A \bowtie^\theta I$ in terms of that of A and I . Moreover, we study the weak amenability of $A \bowtie^\theta I$.

1. INTRODUCTION

Let A and B be Banach algebras, $\theta : A \rightarrow B$ be a continuous Banach algebra homomorphism, which without loss of generality we can assume that $\|\theta\| \leq 1$, and let I be a closed ideal in B . We consider the Banach algebra $A \bowtie^\theta I = \{(a, i) : a \in A, i \in I\}$, the l^1 -direct sum of A and I , with the following product formula:

$$(a, i) \cdot (a', i') = (aa', \theta(a)i' + i\theta(a') + ii').$$

$A \bowtie^\theta I$ is called the *amalgamation of A with B along I with respect to θ* .

The algebraic version of amalgamated Banach algebras are studied by many algebraists, see for example [7, 8, 9, 10, 20].

A special case of amalgamated Banach algebras, with $I = B$, is studied by some authors, see [1, 3, 14], for example. To our knowledge there are no concrete Banach algebra with this structure. While, as Example 2.1 shows, many classes of concrete Banach algebras can be represented as amalgamated Banach algebras.

The organization of paper is as follows. In Section 2 of this paper, after presenting some examples of amalgamated Banach algebras we establish some primary properties of these algebras. In Sections 3 and 4, we characterize the second dual and topological centres of $A \bowtie^\theta I$ as well as its Arens regularity. In Section 5 we characterize the character space of $A \bowtie^\theta I$. Finally, Section 6 is devoted to the investigation of the weak amenability of $A \bowtie^\theta I$.

The first author is supported by University of Tabriz.

2000 *Mathematics Subject Classification.* 46H25, 16E40, 13B02.

Key words and phrases. Banach modules, Topological centre, Weak amenability.

2. SOME EXAMPLES AND PRIMARY PROPERTIES

We commence this section with listing a number of concrete Banach algebras that have the amalgamated structure.

- Example 2.1.** (i) If $\theta = 0$, then $A \bowtie^0 I$ is nothing but the cartesian product of A and I .
(ii) Let A be a non-unital Banach algebra. Then the *unitization* of A , i.e. $A^\# = \mathbb{C} \oplus A$, is the amalgamation of \mathbb{C} with $A^\#$ along A with respect to the homomorphism $\theta : \mathbb{C} \rightarrow A^\#$ defined by $\theta(\lambda) = (\lambda, 0)$.
(iii) Let A be a Banach algebra and X be a Banach A -bimodule. Then the *module extension Banach algebra* $\mathcal{S} = A \oplus X$ is the amalgamation of A with \mathcal{S} along X with respect to the injection $\theta : A \rightarrow \mathcal{S}$ defined by $\theta(a) = (a, 0)$. Notice that the class of module extension Banach algebras include the class of triangular Banach algebras.
(iv) Let A be a Banach algebra and ϕ be a nonzero character on A . Then $A \bowtie^\phi \mathbb{C}$ is the Banach algebra with the underlying Banach space $A \oplus \mathbb{C}$ and with the product

$$(a, \lambda) \cdot (a', \lambda') = (aa', \phi(a)\lambda' + \phi(a')\lambda + \lambda\lambda').$$

- (v) Let A and B be Banach algebras and let ϕ be a nonzero character on A . Then $A \bowtie^\theta B$, the amalgamation of A with $B^\#$ along B with respect to the homomorphism $\theta : A \rightarrow B^\#$ defined by $\theta(a) = (\phi(a), 0)$, is the Banach algebra with the underlying Banach space $A \oplus_1 B$, the l^1 -direct sum of A and B , and with the following product formula:

$$(a, b) \cdot (a', b') = (aa', \phi(a)b' + \phi(a')b + bb').$$

This is a known Banach algebra denoted by $A \oplus_\phi B$, called the ϕ -Lau product of A and B , see [21] for example. This class includes the class of Lau algebras introduced in [17].

- (vi) One of the other interesting examples is the *semidirect product* of Banach algebras. Indeed, let B be a Banach algebra, A be a closed subalgebra of B and I be a closed ideal in B . If $\iota : A \rightarrow B$ is the inclusion map, the amalgamated Banach algebra $C = A \bowtie^\iota I$ is $A \ltimes I$, the semidirect product of A and I [6, Page 8] (as far as we know, the term “semidirect product” in the theory of (commutative) Banach algebras is introduced and studied by Thomas in [22]). We give an important class of Banach algebras which can be recognized as a semidirect product. Let A be a dual Banach algebra with predual A_* and consider A^{**} , the second dual of A equipped with either first or second Arens product (see Section 3 for definitions). It is shown in [6, Theorem 2.15] that $A^{**} = A \ltimes A_*^\perp$, where $A_*^\perp = \{F \in A^{**} : F = 0 \text{ on } A_*\}$. We remark that every von Neumann algebra, the measure algebra $M(G)$ of a locally compact group G , and the second dual of an Arens regular Banach algebra are examples of dual Banach algebras. Also the measure algebra of a locally compact group G has a natural semidirect product structure. In fact we have $M(G) = l^1(G) \ltimes M_c(G)$, where $l^1(G)$ and $M_c(G)$ denote the space of discrete measures and continuous measures in $M(G)$, respectively.

In the following proposition, we have collected some basic properties of the Banach algebra $A \bowtie^\theta I$.

Proposition 2.1. *Let $A \bowtie^\theta I$ be the amalgamation of A with B along I with respect to θ .*

- (i) $A \cong A \times \{0\}$ is a closed subalgebra of $A \bowtie^\theta I$, $I \cong \{0\} \times I$ is a closed ideal in $A \bowtie^\theta I$ and $\frac{A \bowtie^\theta I}{I} \cong A$.
- (ii) $A \bowtie^\theta I$ is commutative if and only if A and $\theta(A) + I$ are commutative.
- (iii) (a, i) is an identity for $A \bowtie^\theta I$ if and only if $a = 1_A$, $i^2 = i$, $i \in \text{Ann}_I(\theta(A))$ and $\theta(a) + i = 1_{\theta(A) + I}$, where $\text{Ann}_I(\theta(A)) = \{j \in I : j\theta(a) = \theta(a)j = 0 \text{ for all } a \in A\}$.
- (iv) $((a_\alpha, i_\alpha))_\alpha$ is a (bounded) left (right, or two-sided) approximate identity for $A \bowtie^\theta I$ if and only if $(a_\alpha)_\alpha$ is a (bounded) left (right, or two-sided) approximate identity for A , $(\theta(a_\alpha) + i_\alpha)_\alpha$ is a (bounded) left (right, or two-sided) approximate identity for $\theta(A) + I$ and $i_\alpha \theta(a) \rightarrow 0$ for all $a \in A$.
- (v) If $A \bowtie^\theta I$ is commutative, then $A \bowtie^\theta I$ is regular if and only if both A and I are regular (see [16, Definition 4.2.1]).
- (vi) $A \bowtie^\theta I$ is amenable if and only if A and I are amenable (see [19, Definition 2.1.9]).

Proof. All of the parts (i)-(iv) can be easily checked. The part (v) follows from Theorems 4.2.6 and 4.3.8 of [16], since $\frac{A \bowtie^\theta I}{I} \cong A$. Finally, (vi) follows from Corollary 2.3.2, Theorem 2.3.7 and Theorem 2.3.10 of [19]. \square

Corollary 2.2. *Let $A \bowtie^{\text{id}} A$ be the amalgamation of A with A along A with respect to the identity map id on A .*

- (i) $A \bowtie^{\text{id}} A$ is commutative if and only if A is commutative.
- (ii) (a, b) is an identity for $A \bowtie^{\text{id}} A$ if and only if $a = 1_A$ and $b = 0$.
- (iii) $((a_\alpha, b_\alpha))_\alpha$ is a (bounded) left (right, or two-sided) approximate identity for $A \bowtie^{\text{id}} A$ if and only if $(a_\alpha)_\alpha$ is a (bounded) left (right, or two-sided) approximate identity for A and $b_\alpha \rightarrow 0$.
- (v) If A is commutative, then $A \bowtie^{\text{id}} A$ is regular if and only if A is regular.
- (vi) $A \bowtie^{\text{id}} A$ is amenable if and only if A is amenable.

3. THE FIRST AND SECOND ARENS PRODUCTS ON $(A \bowtie^\theta I)^{**}$

Let A be a Banach algebra and A^* and A^{**} be the first and second duals of A , respectively. Let $a \in A$, $f \in A^*$. Then $a \cdot f$ and $f \cdot a \in A^*$ are defined by

$$\langle a \cdot f, b \rangle = \langle f, ba \rangle, \quad \langle f \cdot a, b \rangle = \langle f, ab \rangle \quad (b \in A),$$

making A^* an A -bimodule. Similarly, A^{**} is an A -bimodule.

There are two natural products on A^{**} , called the *first and second Arens products*, and are denoted by \square and \diamond , respectively. They were introduced by Arens [2] (for more details the

reader is referred to [4]). We recall briefly the definitions. For $f \in A^*$ and $F \in A^{**}$ define $f \cdot F \in A^*$ and $F \cdot f \in A^*$ by

$$\langle f \cdot F, a \rangle = \langle F, a \cdot f \rangle, \quad \langle F \cdot f, a \rangle = \langle F, f \cdot a \rangle \quad (a \in A).$$

Now, for $F, G \in A^{**}$, define $F \square G \in A^{**}$ and $F \diamond G \in A^{**}$ by

$$\langle F \square G, f \rangle = \langle F, G \cdot f \rangle, \quad \langle F \diamond G, f \rangle = \langle G, f \cdot F \rangle \quad (f \in A^*).$$

Then (A^{**}, \square) and (A^{**}, \diamond) are Banach algebras containing A as a closed subalgebra.

Proposition 3.1. $(A \bowtie^\theta I)^*$ is isometrically isomorphic to $A^* \oplus_\infty I^*$ as Banach spaces. The isomorphism $\Psi : A^* \oplus_\infty I^* \rightarrow (A \bowtie^\theta I)^*$ is given by

$$\langle (a, i), \Psi(f, g) \rangle = f(a) + g(i) \quad ((a, i) \in A \bowtie^\theta I, (f, g) \in A^* \oplus_\infty I^*).$$

Proof. The proof is straightforward and is omitted. \square

Corollary 3.2. $(A \bowtie^\theta I)^{**}$ is isometrically isomorphic to $A^{**} \oplus_1 I^{**}$ as Banach spaces.

Now we explore the left and right module actions of $A \bowtie^\theta I$ on $(A \bowtie^\theta I)^*$ in order to provide a characterization of the first and second Arens product on $(A \bowtie^\theta I)^{**}$.

Theorem 3.3. Let $A \bowtie^\theta I$ be the amalgamation of A with B along I with respect to θ . Then

$$((A \bowtie^\theta I)^{**}, \square) = (A^{**}, \square) \bowtie^{\theta^{**}} (I^{**}, \square),$$

where θ^{**} is the second adjoint of θ .

Proof. Let $a, b \in A$, $i, j \in I$, $f \in A^*$ and $g \in I^*$. Then $(f, g) \cdot (a, i) \in (A \bowtie^\theta I)^*$ can be calculated as follows:

$$\begin{aligned} \langle (b, j), (f, g) \cdot (a, i) \rangle &= \langle (a, i) \cdot (b, j), (f, g) \rangle \\ &= \langle (ab, i\theta(b) + \theta(a)j + ij), (f, g) \rangle \\ &= \langle ab, f \rangle + \langle \theta(a)j, g \rangle + \langle i\theta(b), g \rangle + \langle ij, g \rangle \\ &= \langle b, f \cdot a \rangle + \langle j, g \cdot \theta(a) \rangle + \langle \theta(b), g \cdot i \rangle + \langle j, g \cdot i \rangle \\ &= \langle b, f \cdot a \rangle + \langle j, g \cdot \theta(a) \rangle + \langle b, \theta^*(g \cdot i) \rangle + \langle j, g \cdot i \rangle \\ &= \langle b, f \cdot a + \theta^*(g \cdot i) \rangle + \langle j, g \cdot (\theta(a) + i) \rangle, \end{aligned}$$

and so

$$(3.1) \quad (f, g) \cdot (a, i) = (f \cdot a + \theta^*(g \cdot i), g \cdot (\theta(a) + i)).$$

Further, let $F_1 \in A^{**}$, $F_2 \in I^{**}$. Then, in order to calculate $(F_1, F_2) \cdot (f, g) \in (A \bowtie^\theta I)^*$, one has

$$\begin{aligned}
\langle (a, i), (F_1, F_2) \cdot (f, g) \rangle &= \langle (f, g) \cdot (a, i), (F_1, F_2) \rangle \\
&= \langle (f \cdot a + \theta^*(g \cdot i), g \cdot (\theta(a) + i)), (F_1, F_2) \rangle \\
&= \langle f \cdot a + \theta^*(g \cdot i), F_1 \rangle + \langle g \cdot (\theta(a) + i), F_2 \rangle \\
&= \langle a, F_1 \cdot f \rangle + \langle i, \theta^{**}(F_1) \cdot g \rangle + \langle \theta(a), F_2 \cdot g \rangle + \langle i, F_2 \cdot g \rangle \\
&= \langle a, F_1 \cdot f + \theta^*(F_2 \cdot g) \rangle + \langle i, F_2 \cdot g + \theta^{**}(F_1) \cdot g \rangle \\
&= \langle (a, i), (F_1 \cdot f + \theta^*(F_2 \cdot g), F_2 \cdot g + \theta^{**}(F_1) \cdot g) \rangle.
\end{aligned}$$

Thus

$$(3.2) \quad (F_1, F_2) \cdot (f, g) = (F_1 \cdot f + \theta^*(F_2 \cdot g), F_2 \cdot g + \theta^{**}(F_1) \cdot g).$$

Now for $(F_1, F_2), (G_1, G_2) \in (A \bowtie^\theta I)^{**} \cong A^{**} \oplus_1 I^{**}$ and $(f, g) \in (A \bowtie^\theta I)^* \cong A^* \oplus_\infty I^*$, using (3.1) and (3.2), we have

$$\begin{aligned}
\langle (f, g), (F_1, F_2) \square (G_1, G_2) \rangle &= \langle (G_1, G_2) \cdot (f, g), (F_1, F_2) \rangle \\
&= \langle (G_1 \cdot f + \theta^*(G_2 \cdot g), G_2 \cdot g + \theta^{**}(G_1) \cdot g), (F_1, F_2) \rangle \\
&= \langle G_1 \cdot f + \theta^*(G_2 \cdot g), F_1 \rangle + \langle G_2 \cdot g + \theta^{**}(G_1) \cdot g, F_2 \rangle \\
&= \langle f, F_1 \square G_1 \rangle + \langle G_2 \cdot g, \theta^{**}(F_1) \rangle + \langle g, F_2 \square G_2 \rangle + \langle \theta^{**}(G_1) \cdot g, F_2 \rangle \\
&= \langle f, F_1 \square G_1 \rangle + \langle g, \theta^{**}(F_1) \square G_2 \rangle + \langle g, F_2 \square G_2 \rangle + \langle g, F_2 \square \theta^{**}(G_1) \rangle \\
&= \langle f, F_1 \square G_1 \rangle + \langle g, \theta^{**}(F_1) \square G_2 + F_2 \square G_2 + F_2 \square \theta^{**}(G_1) \rangle \\
&= \langle (f, g), (F_1 \square G_1, \theta^{**}(F_1) \square G_2 + F_2 \square G_2 + F_2 \square \theta^{**}(G_1)) \rangle.
\end{aligned}$$

Therefore,

$$(F_1, F_2) \square (G_1, G_2) = (F_1 \square G_1, \theta^{**}(F_1) \square G_2 + F_2 \square G_2 + F_2 \square \theta^{**}(G_1)).$$

This completes the proof. \square

Similarly, as notation in the proof of Theorem 3.3, one can show that

$$(3.3) \quad (a, i) \cdot (f, g) = (a \cdot f + \theta^*(i \cdot g), (\theta(a) + i) \cdot g),$$

$$(f, g) \cdot (F_1, F_2) = (f \cdot F_1 + \theta^*(g \cdot F_2), g \cdot F_2 + g \cdot \theta^{**}(F_1)),$$

$$(F_1, F_2) \diamond (G_1, G_2) = (F_1 \diamond G_1, \theta^{**}(F_1) \diamond G_2 + F_2 \diamond \theta^{**}(G_1) + F_2 \diamond G_2).$$

Therefore,

$$((A \bowtie^\theta I)^{**}, \diamond) = (A^{**}, \diamond) \bowtie^{\theta^{**}} (I^{**}, \diamond).$$

4. TOPOLOGICAL CENTRES

Let A be a Banach algebra and X a Banach A -bimodule. Then X^{**} is canonically an (A^{**}, \square) -bimodule ((A^{**}, \diamond) -bimodule), see [4, Page 248]. Let $x'' \in X^{**}$ and let $L_{x''}, R_{x''} : (A^{**}, \square) \rightarrow X^{**}$ be the left and right multiplication operators, respectively, i.e.

$$L_{x''}(a'') = x'' \square a'' = \lim_{\beta} \lim_{\alpha} x_{\beta} a_{\alpha} \quad \text{and} \quad R_{x''}(a'') = a'' \square x'' = \lim_{\alpha} \lim_{\beta} a_{\alpha} x_{\beta} \quad (a'' \in A^{**}),$$

where (a_{α}) and (x_{β}) are nets in A^{**} and X^{**} , respectively, in such a way that $a_{\alpha} \rightarrow a''$ in the w^* -topology of A^{**} and $x_{\beta} \rightarrow x''$ in the w^* -topology of X^{**} .

Likewise let $\mathcal{L}_{x''}, \mathcal{R}_{x''} : (A^{**}, \diamond) \rightarrow X^{**}$ be the left and right multiplication operators, respectively, i.e.

$$\mathcal{L}_{x''}(a'') = x'' \diamond a'' = \lim_{\alpha} \lim_{\beta} x_{\beta} a_{\alpha} \quad \text{and} \quad \mathcal{R}_{x''}(a'') = a'' \diamond x'' = \lim_{\beta} \lim_{\alpha} a_{\alpha} x_{\beta} \quad (a'' \in A^{**}).$$

The left and right topological centres, $Z_A^{\ell,t}(X^{**})$ and $Z_A^{r,t}(X^{**})$ of X^{**} are

$$Z_A^{\ell,t}(X^{**}) = \{x'' \in X^{**} : L_{x''} = \mathcal{L}_{x''}\} = \{x'' \in X^{**} : x'' \square a'' = x'' \diamond a'', \forall a'' \in A^{**}\},$$

and

$$Z_A^{r,t}(X^{**}) = \{x'' \in X^{**} : R_{x''} = \mathcal{R}_{x''}\} = \{x'' \in X^{**} : a'' \square x'' = a'' \diamond x'', \forall a'' \in A^{**}\},$$

respectively. Then we say that X is *Arens regular* (as an A -bimodule) or A *acts regularly on X* if

$$Z_A^{\ell,t}(X^{**}) = Z_A^{r,t}(X^{**}) = X^{**},$$

and X is *left strongly Arens irregular* if $Z_A^{\ell,t}(X^{**}) = X$, *right strongly Arens irregular* if $Z_A^{r,t}(X^{**}) = X$, and *strongly Arens irregular* if it is both left and right strongly Arens irregular.

If $X = A$, we will use the common notation $Z_t^{\ell}(A^{**})$ and $Z_t^r(A^{**})$ in place of $Z_A^{\ell,t}(A^{**})$ and $Z_A^{r,t}(A^{**})$, respectively.

Now, let B be a Banach algebra, I be a closed ideal in B and $\theta : A \rightarrow B$ be a continuous Banach algebra homomorphism. Then we define

$$Z_{\theta^{**}}^{\ell}(A^{**}) = \{F \in Z_t^{\ell}(A^{**}) : \theta^{**}(F) \in Z_I^{\ell}(\theta(A)^{**})\},$$

note that $\theta(A)^{**} = \theta^{**}(A^{**})$ ([4, Page 251]), where

$$Z_I^{\ell}(\theta(A)^{**}) = \{F \in \theta(A)^{**} : L_F = \mathcal{L}_F \text{ on } I^{**}\} = \{F \in \theta(A)^{**} : F \square G = F \diamond G, \forall G \in I^{**}\}.$$

Theorem 4.1. *With above notation and assumptions, one has*

$$Z_t^{\ell}((A \bowtie^{\theta} I)^{**}) = Z_t^{\ell}(A^{**} \bowtie^{\theta^{**}} I^{**}) = Z_{\theta^{**}}^{\ell}(A^{**}) \bowtie^{\theta^{**}} (Z_t^{\ell}(I^{**}) \cap Z_{\theta(A)}^{\ell,t}(I^{**})).$$

Proof. Let $(F_1, G_1) \in Z_t^\ell(A^{**} \bowtie^{\theta^{**}} I^{**})$. Then $L_{(F_1, G_1)} = \mathcal{L}_{(F_1, G_1)}$ if and only if for all $(F_2, G_2) \in A^{**} \bowtie^{\theta^{**}} I^{**}$,

$$(F_1, G_1) \square (F_2, G_2) = (F_1, G_1) \diamond (F_2, G_2),$$

if and only if

$$(F_1 \square F_2, \theta^{**}(F_1) \square G_2 + G_1 \square \theta^{**}(F_2) + G_1 \square G_2) = (F_1 \diamond F_2, \theta^{**}(F_1) \diamond G_2 + G_1 \diamond \theta^{**}(F_2) + G_1 \diamond G_2),$$

if and only if $L_{F_1} = \mathcal{L}_{F_1}$, $L_{\theta^{**}(F_1)} = \mathcal{L}_{\theta^{**}(F_1)}$ on I^{**} , $L_{G_1} = \mathcal{L}_{G_1}$ on I^{**} and $\theta(A)^{**}$. Hence,

$$(F_1, G_1) \in Z_{\theta^{**}}^\ell(A^{**}) \bowtie^{\theta^{**}} (Z_t^\ell(I^{**}) \cap Z_{\theta(A)}^{\ell, t}(I^{**})).$$

□

Using above theorem we characterize Arens regularity and strong Arens irregularity of $A \bowtie^\theta I$.

Corollary 4.2. *$A \bowtie^\theta I$ is Arens regular if and only if A and I are Arens regular, $\theta(A)$ acts regularly on I and I acts regularly on $\theta(A)$.*

Corollary 4.3. *$A \bowtie^\theta I$ is strongly Arens irregular if and only if A and I are strongly Arens irregular, I acts strongly irregular on $\theta(A)$ and $\theta(A)$ acts strongly irregular on I .*

In the following example we determine topological centres of some amalgamated Banach algebras.

Example 4.1. Keep the notation of Example 2.1.

(i) If $\theta = 0$, then

$$Z_t^\ell((A \oplus I)^{**}) = Z_t^\ell((A \bowtie^0 I)^{**}) = Z_t^\ell(A^{**}) \bowtie^0 Z_t^\ell(I^{**}) = Z_t^\ell(A^{**}) \oplus Z_t^\ell(I^{**}).$$

(ii) $Z_t^\ell((A^\#)^{**}) = \mathbb{C} \oplus Z_t^\ell(A^{**}) = Z_t^\ell(A^{**})^\#$.

(iii) ([11]) Let $\mathcal{S} = A \oplus X$ be the module extension Banach algebra corresponding A and X . Then, by noting that θ is the canonical embedding of A into \mathcal{S} , and $I = X$ with $X^2 = 0$, we have

$$Z_{\theta^{**}}^\ell(A^{**}) = \{F \in Z_t^\ell(A^{**}) : F \in Z_X^\ell(A^{**})\} = Z_t^\ell(A^{**}) \cap Z_X^\ell(A^{**}),$$

and

$$Z_t^\ell(I^{**}) \cap Z_{\theta(A)}^{\ell, t}(I^{**}) = Z_t^\ell(X^{**}) \cap Z_A^{\ell, t}(X^{**}) = Z_t^\ell(X^{**}) = X^{**}.$$

Therefore,

$$Z_t^\ell(\mathcal{S}^{**}) = (Z_t^\ell(A^{**}) \cap Z_X^\ell(A^{**})) \bowtie^{\theta^{**}} X^{**},$$

that is, $Z_t^\ell(\mathcal{S}^{**})$ is the module extension Banach algebra corresponding $Z_t^\ell(A^{**}) \cap Z_X^\ell(A^{**})$ and X^{**} .

(iv) $Z_t^\ell((A \bowtie^\phi \mathbb{C})^{**}) = Z_t^\ell(A^{**}) \bowtie^\phi \mathbb{C}$. Details are similar to details of the next general case.

- (v) ([21, Corollary 2.13]) For computing $Z_t^\ell((A \oplus_\phi B)^{**})$ we note that since $\theta : A \rightarrow B^\#$ is defined by $\theta(a) = (\phi(a), 0) = \phi(a)$, one can easily check that $\theta^{**} : A^{**} \rightarrow (B^\#)^{**} = \mathbb{C} \oplus B^{**}$ is given by $\theta^{**}(F) = (F(\phi), 0) = F(\phi) = \phi(F)$. So

$$Z_{\theta^{**}}^\ell(A^{**}) = Z_\phi^\ell(A^{**}) = Z_t^\ell(A^{**}),$$

and

$$Z_{\theta(A)}^{\ell,t}(I^{**}) = Z_{\mathbb{C}}^{\ell,t}(B^{**}) = B^{**}.$$

Whence

$$Z_t^\ell((A \oplus_\phi B)^{**}) = Z_t^\ell(A^{**}) \bowtie^{\theta^{**}} Z_t^\ell(B^{**}) = Z_t^\ell(A^{**}) \oplus_\phi Z_t^\ell(B^{**}).$$

- (vi) If $I = B$ and θ is surjective, then

$$Z_t^\ell((A \bowtie^\theta B)^{**}) = (Z_t^\ell(A^{**}) \cap (\theta^{**})^{-1}(Z_t^\ell(B^{**}))) \bowtie^{\text{id}} Z_t^\ell(B^{**}).$$

- (vii) Assume that $B = A$. Then

$$Z_t^\ell((A \bowtie^{\text{id}} I)^{**}) = (Z_t^\ell(A^{**}) \cap Z_I^\ell(A^{**})) \bowtie^{\text{id}} (Z_t^\ell(I^{**}) \cap Z_A^{\ell,t}(I^{**})).$$

- (ix) $Z_t^\ell((A \bowtie^{\text{id}} A)^{**}) = Z_t^\ell(A^{**}) \bowtie^{\text{id}} Z_t^\ell(A^{**})$.

- (x) Let $B = I = A^{**}$ and let $\iota : A \rightarrow A^{**}$ be the cononical injection. Then

$$Z_{A^{**}}^\ell(\iota(A)^{**}) = Z_{A^{**}}^\ell(A^{**}) = A^{**},$$

and so $Z_{\iota^{**}}^\ell(A^{**}) = Z_t^\ell(A^{**})$. Also $Z_{\iota(A)}^{\ell,t}((A^{**})^{**}) = Z_A^{\ell,t}(A^{****}) = A^{****}$, and thus

$$Z_t^\ell((A \bowtie^\iota A^{**})^{**}) = Z_t^\ell(A^{**}) \bowtie^{\iota^{**}} Z_t^\ell(A^{****}).$$

Example 4.2. By Example 4.1 in mind we have the followings:

- (i) The Banach algebra $A \bowtie^0 I$ is Arens regular if and only if A and I are Arens regular.
- (ii) The unitization of A , $A^\#$, is Arens regular if and only if A is Arens regular.
- (iii) The module extension Banach algebra $\mathcal{S} = A \oplus X$ is Arens regular if and only if A is Arens regular and A acts regularly on X .
- (iv) $A \bowtie^\phi \mathbb{C}$ is Arens regular if and only if A is Arens regular.
- (v) $A \oplus_\phi B$ is Arens regular if and only if A and B are Arens regular.
- (vi) If $I = B$ and θ is surjective, then $A \bowtie^\theta B$ is Arens regular if and only if A and B are Arens regular.
- (vii) If $B = A$, then $A \bowtie^\theta I$ is Arens regular if and only if A is Arens regular and A and I act regularly on each other.
- (ix) The Banach algebra $A \bowtie^{\text{id}} A$ is Arens regular if and only if A is Arens regular.
- (x) The Banach algebra $A \bowtie^\iota A^{**}$ is Arens regular if and only if A and A^{**} are Arens regular.

Example 4.3. Let G be an infinite locally compact group. Then by [18, Theorem 1] we have $Z_t^\ell(L^1(G)^{**}) = L^1(G)$, and so

$$Z_t^\ell((L^1(G) \rtimes^{\text{id}} L^1(G))^{**}) = Z_t^\ell(L^1(G)^{**}) \rtimes^{\text{id}} Z_t^\ell(L^1(G)^{**}) = L^1(G) \rtimes^{\text{id}} L^1(G).$$

Therefore, $L^1(G) \rtimes^{\text{id}} L^1(G)$ is strongly Arens irregular.

5. CHARACTERS OF $A \rtimes^\theta I$

In this section, first, we provide a characterization of character space of $A \rtimes^\theta I$, and then we calculate the Jacobson radical of $A \rtimes^\theta I$ when it is commutative.

Theorem 5.1. *Let $\sigma(A) \neq \emptyset$ and $\overline{\theta(A)I \cup I\theta(A)} = I$. Then $\sigma(A \rtimes^\theta I) = E \cup F$, where*

$$E = \{((i \cdot \psi) \circ \theta, \psi) : \psi \in \sigma(I), i \in I, \psi(i) = 1\}, \quad F = \{(\phi, 0) : \phi \in \sigma(A)\}.$$

Moreover, E is open and F is closed in $\sigma(A \rtimes^\theta I)$.

Proof. Let $(\phi, \psi) \in \sigma(A \rtimes^\theta I)$ and $(a, i), (a', i') \in A \rtimes^\theta I$. Then

$$\begin{aligned} \phi(aa') + \psi(\theta(a)i' + i\theta(a') + ii') &= \langle (\phi, \psi), (aa', \theta(a)i' + i\theta(a') + ii') \rangle \\ &= \langle (\phi, \psi), ((a, i) \cdot (a', i')) \rangle \\ (5.1) \quad &= (\phi(a) + \psi(i))(\phi(a') + \psi(i')) \\ &= \phi(a)\phi(a') + \phi(a)\psi(i') + \psi(i)\phi(a') + \psi(i)\psi(i'). \end{aligned}$$

By taking $a = a' = 0$ we see that $\psi \in \sigma(I) \cup \{0\}$. Next by taking $i = i' = 0$ it follows that $\phi \in \sigma(A) \cup \{0\}$. But from (5.1), $\phi = 0$ implies $\psi(\theta(a)i') + \psi(i\theta(a')) = 0$ for all $a, a' \in A$ and $i, i' \in I$, from which it follows that $\psi = 0$ on $\theta(A)I \cup I\theta(A)$, and hence $\psi = 0$. But this is a contradiction since $(\phi, \psi) \in \sigma(A \rtimes^\theta I)$ and so $(\phi, \psi) \neq (0, 0)$. Now we have two cases:

Case I: If $\psi = 0$, then $(\phi, \psi) = (\phi, 0)$.

Case II: If $\psi \neq 0$, then by (5.1) we have

$$\psi(\theta(a)i') + \psi(i\theta(a')) - \psi(i)\phi(a') - \phi(a)\psi(i') = 0,$$

which implies (take $a' = 0, i = 0$),

$$\psi(\theta(a)i') = \phi(a)\psi(i') \quad (a \in A, i' \in I).$$

Choose $i' \in I$ such that $\psi(i') = 1$, then $\phi(a) = \psi(\theta(a)i') = (i' \cdot \psi) \circ \theta(a)$ for all $a \in A$. Therefore, $(\phi, \psi) = ((i' \cdot \psi) \circ \theta, \psi)$ with $\psi(i') = 1$.

Since the reverse inclusion is easy to check, so we omit its proof.

Now we show that E is open in the w^* -topology of $\sigma(A \rtimes^\theta I)$ induced from w^* -topology of $A^* \times I^*$. Let $((i \cdot \psi_0) \circ \theta, \psi_0) \in \sigma(A \rtimes^\theta I)$. Then there is $i_0 \in I$ in such a way that $\psi(i_0) \neq 0$. Let $\varepsilon = |\psi(i_0)|$. Then

$$\begin{aligned} U &= \{(\phi, \psi) \in \sigma(A \rtimes^\theta I) : |(\phi, \psi)(0, i_0) - ((i \cdot \psi_0) \circ \theta, \psi_0)(0, i_0)| < \varepsilon\} \\ &= \{(\phi, \psi) \in \sigma(A \rtimes^\theta I) : |\psi(i_0) - \psi_0(i_0)| < \varepsilon\}, \end{aligned}$$

is a neighborhood of $((i \cdot \psi_0) \circ \theta, \psi_0)$ in the w^* -topology of $\sigma(A \bowtie^\theta I)$. Since $(\phi, 0) \in U$ leads to the contradiction $|\psi_0(i_0)| < \varepsilon$, it follows that $U \subseteq E$. Therefore, E is open and F is closed. \square

Corollary 5.2. ([21, Proposition 2.4]) *Let $\sigma(A) \neq \emptyset$ and $\phi \in \sigma(A)$. Then*

$$\sigma(A \oplus_\phi B) = \{(\varphi, 0) : \varphi \in \sigma(A)\} \cup \{(\phi, \psi) : \psi \in \sigma(B)\} = (\sigma(A) \times \{0\}) \cup (\{\phi\} \times \sigma(B)).$$

Proof. It is enough to note that $(i \cdot \psi) \circ \theta = \theta$ if $\psi(i) = 1$. \square

Let A be a commutative Banach algebra. The *radical* of A , $\text{rad } A$, is the intersection of the kernels of all characters of A . Also A is called *semisimple* if $\text{rad } A = \{0\}$.

Theorem 5.3. *Let $A \bowtie^\theta I$ be commutative, $\sigma(A) \neq \emptyset$ and $\overline{\theta(A)I} = I$. Then $\text{rad}(A \bowtie^\theta I) = \text{rad } A \oplus \text{rad } I$.*

Proof. Let $(a, i) \in \text{rad}(A \bowtie^\theta I)$. Then for each $\phi \in \sigma(A)$, $\phi(a) = (\phi, 0)(a, i) = 0$, that is, $a \in \text{rad } A$. Now let $\psi \in \sigma(I)$ and $\psi(j) = 1$ for some $j \in I$. Then $(j \cdot \psi) \circ \theta$ belongs to $\sigma(A)$ and so $(j \cdot \psi) \circ \theta(a) = 0$. Hence

$$\psi(i) = ((j \cdot \psi) \circ \theta, \psi)(a, i) - (j \cdot \psi) \circ \theta(a) = 0,$$

and thus $i \in \text{rad } I$.

Conversely let $a \in \text{rad } A$ and $i \in \text{rad } I$. Then for each $\phi \in \sigma(A)$, $(\phi, 0)(a, i) = \phi(a) = 0$ and for each $\psi \in \sigma(I)$,

$$((j \cdot \psi) \circ \theta, \psi)(a, i) = (j \cdot \psi) \circ \theta(a) + \psi(i) = 0.$$

Therefore, by Theorem 5.1, $(a, i) \in \text{rad}(A \bowtie^\theta I)$. \square

Corollary 5.4. *Let $A \bowtie^\theta I$ be commutative, $\sigma(A) \neq \emptyset$ and $\overline{\theta(A)I} = I$. Then $A \bowtie^\theta I$ is semisimple if and only if both A and I are semisimple.*

6. WEAK AMENABILITY

Let A be a Banach algebra and X a Banach A -bimodule. A *derivation* from A into X is a bounded linear map satisfying $D(ab) = a \cdot D(b) + D(a) \cdot b$ for all $a, b \in A$. For each $x \in X$ we denote by ad_x the derivation $D(a) = a \cdot x - x \cdot a$ for all $a \in A$, called an *inner derivation*. We denote by $\mathcal{Z}^1(A, X)$ the space of all derivations from A into X and by $\mathcal{B}^1(A, X)$ the space of all inner derivations from A into X . The *first cohomology group* of A with coefficients in X is $\mathcal{H}^1(A, X) = \mathcal{Z}^1(A, X) / \mathcal{B}^1(A, X)$. A Banach algebra A is called *weakly amenable* if $\mathcal{H}^1(A, A^*) = 0$.

In [12], B. E. Forrest and L. W. Marcoux have investigated the weak amenability of triangular Banach algebras, and Y. Zhang has studied the weak amenability of module extension Banach algebras [23]. Motivated by these earlier investigations, in this section, we study the weak amenability of amalgamated Banach algebra $A \bowtie^\theta I$.

Theorem 6.1. *If $A \bowtie^\theta I$ is commutative, then $A \bowtie^\theta I$ is weakly amenable if and only if A and I are weakly amenable.*

Proof. This is immediate by Proposition 2.1, [4, Propositions 2.8.64 and 2.8.65(ii)] and Theorem 2.8.69(i)] and noting that $\frac{A \bowtie^\theta I}{I} \cong A$. \square

Example 6.1. Let G be a locally compact abelian group and consider $M(G) = l^1(G) \ltimes M_c(G)$, the semidirect product of $l^1(G)$ and $M_c(G)$; see Example 2.1(iv). Since $l^1(G)$ is weakly amenable, by above theorem, $M(G)$ is weakly amenable if and only if $M_c(G)$ is weakly amenable.

In general case, we have one direction of Theorem 6.1.

Proposition 6.2. *If A and I are weakly amenable, then $A \bowtie^\theta I$ is also weakly amenable.*

Proof. It follows immediately from Proposition 2.1 and [4, Proposition 2.8.65(ii)]. \square

The converse of above proposition does not hold in general. Indeed, it is shown in [15] that the augmentation ideal I of $L^1(SL(2, \mathbb{R}))$ is not weakly amenable and that its unitization $I^\#$ is weakly amenable.

Example 6.2. Let G be a locally compact group. Since $M(G) = l^1(G) \ltimes M_c(G)$ and $l^1(G)$ is always weakly amenable, by Proposition 6.2, $M(G)$ is weakly amenable, provided that $M_c(G)$ is weakly amenable.

Example 6.3. Let G be a locally compact group. Then $l^1(G) \ltimes L^1(G)$ is weakly amenable.

In order to prove a partial converse of Proposition 6.2 we first look at derivations from A to A^* .

Proposition 6.3. $\mathcal{H}^1(A, A^*)$ embeds in $\mathcal{H}^1(A \bowtie^\theta I, (A \bowtie^\theta I)^*)$.

Proof. Every $D \in \mathcal{Z}^1(A, A^*)$ defines a derivation $\tilde{D} : A \bowtie^\theta I \rightarrow (A \bowtie^\theta I)^*$ by $\tilde{D}(a, i) = (D(a), 0)$, and it can be easily checked that \tilde{D} is inner if and only if D is inner. It follows that the mapping $D \mapsto \tilde{D}$ induces an embedding from $\mathcal{H}^1(A, A^*)$ into $\mathcal{H}^1(A \bowtie^\theta I, (A \bowtie^\theta I)^*)$. \square

Corollary 6.4. *If $A \bowtie^\theta I$ is weakly amenable, then so is A .*

The following corollary has been obtained in [13] with a different method. In fact, we have given a short proof for this result.

Corollary 6.5. ([13, Theorem 2.2]) *Let A be a dual Banach algebra. If A^{**} is weakly amenable, then so is A .*

Weak amenability of module extension Banach algebras is extensively studied in [23]. We are going to characterize the weak amenability of Banach algebras $A \bowtie^{\text{id}} A$ and $A \oplus_\phi B$.

6.1. Weak Amenability of $A \bowtie^{\text{id}} A$. Here, we focus on the special case $A \bowtie^{\text{id}} A$.

Proposition 6.6. *Let A^2 be dense in A . Then $D \in \mathcal{Z}^1(A \bowtie^{\text{id}} A, (A \bowtie^{\text{id}} A)^*)$ if and only if*

$$D(a, b) = (D_1(a) + D_2(b), D_2(a) + D_2(b)) \quad (a, b \in A),$$

for some $D_1, D_2 \in \mathcal{Z}^1(A, A^)$. Moreover, $D = \text{ad}_{(f, g)}$ if and only if $D_1 = \text{ad}_f$ and $D_2 = \text{ad}_g$, where $f, g \in A^*$.*

Proof. Let $D : A \bowtie^{\text{id}} A \rightarrow (A \bowtie^{\text{id}} A)^*$ be a derivation. Then we may write

$$D(a, b) = (D_1(a) + D_2(b), D_3(a) + D_4(b)) \quad (a, b \in A),$$

where $D_k : A \rightarrow A^*$ ($1 \leq k \leq 4$) is a linear operator. If we use the derivation property of D together with the equations (3.1) and (3.3), we get

$$\begin{aligned} & (D_1(a_1 a_2) + D_2(a_1 b_2 + b_1 a_2 + b_1 b_2), D_3(a_1 a_2) + D_4(a_1 b_2 + b_1 a_2 + b_1 b_2)) = \\ & (a_1 D_1(a_2) + a_1 D_2(b_2) + b_1 D_3(a_2) + b_1 D_4(b_2) + D_1(a_1) a_2 + D_2(b_1) a_2 + D_3(a_1) b_2 + D_4(b_1) b_2, \\ & (a_1 [D_3(a_2) + D_4(b_2)] + b_1 [D_3(a_2) + D_4(b_2)] + [D_3(a_1) + D_4(b_1)] a_2 + [D_3(a_1) + D_4(b_1)] b_2). \end{aligned}$$

By setting $a_1 = a_2 = 0$, we see that $D_1, D_3 \in \mathcal{Z}^1(A, A^*)$. By setting $b_1 = b_2 = 0$ and noting that A^2 is dense in A , we get $D_2 = D_4 \in \mathcal{Z}^1(A, A^*)$. Also, by choosing $a_2 = b_1 = 0$, we obtain

$$D_2(a_1 b_2) = a_1 \cdot D_2(b_2) + D_3(a_1) \cdot b_2,$$

which implies $D_2(a_1) \cdot b_2 = D_3(a_1) \cdot b_2$ for all $a_1, b_2 \in A$. Since A^2 is dense in A , it follows that $D_2 = D_3$. The claim about inner derivations can be easily verified. \square

Theorem 6.7. *Let A^2 be dense in A . Then, as vector spaces, we have*

$$\mathcal{H}^1(A \bowtie^{\text{id}} A, (A \bowtie^{\text{id}} A)^*) \cong \mathcal{H}^1(A, A^*) \oplus \mathcal{H}^1(A, A^*).$$

Proof. Define $\varphi : \mathcal{Z}^1(A, A^*) \oplus \mathcal{Z}^1(A, A^*) \rightarrow \mathcal{Z}^1(A \bowtie^{\text{id}} A, (A \bowtie^{\text{id}} A)^*)$ by $\varphi(D_1, D_2) = D$, where

$$D(a, b) = (D_1(a) + D_2(b), D_2(a) + D_2(b)) \quad (a, b \in A).$$

The Proposition 6.6 shows that φ is well defined and onto. Since D is inner if and only if D_1 and D_2 are inner, according to the Proposition 6.6, φ induces the desired isomorphism. \square

Since every weakly amenable Banach algebra is square dense, we have the following corollary.

Corollary 6.8. *The Banach algebra $A \bowtie^{\text{id}} A$ is weakly amenable if and only if A is weakly amenable.*

Example 6.4. Let A be a C^* -algebra or a group algebra of a locally compact group. Then $A \bowtie^{\text{id}} A$ is weakly amenable.

6.2. Weak Amenability of $A \oplus_\phi B$. Let $(a, b) \in A \oplus_\phi B$ and $(f, g) \in (A \oplus_\phi B)^*$. Then $(a, b) \cdot (f, g), (f, g) \cdot (a, b) \in (A \oplus_\phi B)^*$ are given by

$$(6.1) \quad (a, b) \cdot (f, g) = (a \cdot f + g(b)\phi, \phi(a)g + b \cdot g),$$

$$(6.2) \quad (f, g) \cdot (a, b) = (f \cdot a + g(b)\phi, \phi(a)g + g \cdot b).$$

Proposition 6.9. *Let B^2 be dense in B . Then $D \in \mathcal{Z}^1(A \oplus_\phi B, (A \oplus_\phi B)^*)$ if and only if*

$$D(a, b) = (D_1(a) + D_2(b), D_4(b)) \quad (a \in A, b \in B),$$

such that

- (i) $D_1 \in \mathcal{Z}^1(A, A^*)$,
- (ii) $D_4 \in \mathcal{Z}^1(B, B^*)$,
- (iii) $D_2 : B \rightarrow A$ is a bounded linear map satisfying
 - (1) $a \cdot D_2(b) = D_2(b) \cdot a = \phi(a)D_2(b)$ for all $a \in A$ and $b \in B$,
 - (2) $D_2(bb') = \langle b, D_4(b') \rangle \phi + \langle b', D_4(b) \rangle \phi$ for all $b, b' \in B$.

Moreover, $D = \text{ad}_{(f,g)}$ if and only if $D_1 = \text{ad}_f$, $D_2 = 0$ and $D_4 = \text{ad}_g$ ($f \in A^*, g \in B^*$).

Proof. Let $D : A \oplus_\phi B \rightarrow (A \oplus_\phi B)^* \cong A^* \oplus_\infty B^*$ be a derivation. Then D is of the form

$$D(a, b) = (D_1(a) + D_2(b), D_3(a) + D_4(b)) \quad (a \in A, b \in B),$$

where $D_1 : A \rightarrow A^*$, $D_2 : B \rightarrow A^*$, $D_3 : A \rightarrow B^*$ and $D_4 : B \rightarrow B^*$ are linear operators. If we use the derivation property of D together with the equations (6.1) and (6.2), we get

$$\begin{aligned} & (D_1(aa') + \phi(a)D_2(b') + \phi(a')D_2(b) + D_2(bb'), D_3(aa') + \phi(a)D_4(b') + \phi(a')D_4(b) + D_4(bb')) = \\ & (aD_1(a') + aD_2(b') + \langle b, D_3(a') \rangle \phi + \langle b, D_4(b') \rangle \phi, \phi(a)D_3(a') + \phi(a)D_4(b') + bD_3(a') + bD_4(b')) + \\ & (D_1(a)a' + D_2(b)a' + \langle b', D_3(a) \rangle \phi + \langle b', D_4(b) \rangle \phi, \phi(a')D_3(a) + \phi(a')D_4(b)) + D_3(a)b' + D_4(b)b'. \end{aligned}$$

By setting $b = b' = 0$ we see that $D_1 \in \mathcal{Z}^1(A, A^*)$ and $D_3 \in \mathcal{Z}_\phi^1(A, A^*)$. Letting $a = a' = 0$ one obtains $D_4 \in \mathcal{Z}^1(B, B^*)$ and $D_2(bb') = \langle b, D_4(b') \rangle \phi + \langle b', D_4(b) \rangle \phi$.

Now put $a = b' = 0$. Then we get $b \cdot D_3(a') = 0$ in B^* which implies $D_3 = 0$ by density of B^2 in B . Hence $D_2(b) \cdot a' = \phi(a')D_2(b)$. Similarly, choosing $a' = b = 0$ gives $a \cdot D_2(b') = \phi(a)D_2(b')$.

Using (6.1) and (6.2) one can easily see that $D = \text{ad}_{(f,g)}$ if and only if $D_1 = \text{ad}_f$, $D_2 = 0$ and $D_4 = \text{ad}_g$. \square

Let B be a Banach algebra. A derivation $D : B \rightarrow B^*$ is called cyclic if

$$\langle b, D(b') \rangle + \langle b', D(b) \rangle = 0 \quad \text{for all } b, b' \in B.$$

We denote by $\mathcal{Z}_c^1(B, B^*)$ the space of all cyclic derivations which includes $\mathcal{B}^1(B, B^*)$. The first cyclic cohomology group of B is $\mathcal{H}_c^1(B, B^*) = \mathcal{Z}_c^1(B, B^*) / \mathcal{B}^1(B, B^*)$.

Theorem 6.10. $\mathcal{H}^1(A, A^*) \oplus \mathcal{H}_c^1(B, B^*)$ embeds in $\mathcal{H}^1(A \oplus_\phi B, (A \oplus_\phi B)^*)$.

Proof. Define $\psi : \mathcal{Z}^1(A, A^*) \oplus \mathcal{Z}_c^1(B, B^*) \longrightarrow \mathcal{Z}^1(A \oplus_\phi B, (A \oplus_\phi B)^*)$ by $\psi(D_1, D_2) = D$, where

$$D(a, b) = (D_1(a), D_2(b)) \quad (a \in A, b \in B).$$

It follows from Proposition 6.9 that D is a derivation and it is inner if and only if D_1 and D_2 are inner. So ψ induces an injective linear map from $\mathcal{H}^1(A, A^*) \oplus \mathcal{H}_c^1(B, B^*)$ into $\mathcal{H}^1(A \oplus_\phi B, (A \oplus_\phi B)^*)$. \square

Corollary 5.6 of [15] shows that in general $\mathcal{H}^1(B, B^*)$ does not embeds into $\mathcal{H}^1(A \oplus_\phi B, (A \oplus_\phi B)^*)$, and thus it seems that Theorem 6.10 be the best that one could expect.

Corollary 6.11. ([21, Theorem 2.11]) *If $A \oplus_\phi B$ is weakly amenable, then A is weakly amenable and B is cyclicly amenable.*

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